FACTORS OF GIBBS MEASURES FOR FULL SHIFTS

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ABSTRACT. We study the images of Gibbs measures under one block factor maps on full shifts, and the changes in the variations of the corresponding potential functions.

1. Introduction

In this note we consider the images of invariant measures under simple factor maps. Let $\sigma_1 : \Sigma_1 \to \Sigma_1$ and $\sigma_2 : \Sigma_2 \to \Sigma_2$ denote two full shifts, on finite alphabets A and B, respectively. Assume that $\Pi : \Sigma_1 \to \Sigma_2$ is a one block factor map (i.e., a semi-conjugacy satisfying $\Pi\sigma_1 = \sigma_2\Pi$ where $(\Pi(x))_0 = \pi(x_0)$ for a surjective map $\pi : A \to B$). Furthermore, let us consider a σ_1-invariant probability measure μ_{ψ_1} which is a Gibbs measure for a potential function $\psi_1 : \Sigma_1 \to \mathbb{R}$. In particular, the image $\nu = \pi_*\mu_{\psi_1}$ will be a σ_2-invariant probability measure.

A very natural question would be to ask under which hypotheses on ψ_1 would ν be a Gibbs measure for a potential $\psi_2 : \Sigma_2 \to \mathbb{R}$ (i.e., $\nu = \mu_{\psi_2}$), and how the regularity of ψ_1 is reflected in that of ψ_2.

In the particular case that ψ_1 is locally constant, and thus μ_{ψ_1} is a generalized Markov measure, it was shown by Chazottes and Ugalde in (2) that ν is a Gibbs measure for a Hölder continuous function $\psi_2 : \Sigma_2 \to \mathbb{R}$ (i.e., $\nu = \mu_{\psi_2}$), although not necessarily still a generalized Markov measure. One of our main results is the following.

Theorem 1.1. *Assume that* $\sum_{n=0}^{\infty} n var_n(\psi_1) < +\infty$ *then* $\sum_{n=0}^{\infty} var_n(\psi_2) < +\infty$.

The method we describe can be extended to the case of suitable subshifts of finite type and factor maps. These results will appear in a separate paper by the first author.

In section 2 we recall some basic properties of Gibbs measures. In section 3, we discuss a construction of the potential function ψ_2. In section 4 we present the proof of a key step in this construction (Proposition 3.2). Finally, in section 5 we present and prove the main results.

After completing this work, Ugalde informed the authors of his contemporaneous work with Chazottes. In (3) they have proved related results using a somewhat different method. In particular, in (3) it is shown that if $\sum_{n=0}^{\infty} n^{2+\epsilon}\mathsf{var}_n(\psi_1) < +\infty$ then $\sum_{n=0}^{\infty} \mathsf{var}_n(\psi_2) < +\infty$. However, the methods there appear to give sharper bounds on the actual rates of decay of the terms $\mathsf{var}_n(\psi_2)$ as $n \to +\infty$ than we can obtain.

2. Gibbs measures and Equilibrium states

We begin with some general definitions and results. Given $k \geq 2$, let $\Sigma = \{1, \cdots, k\}^{\mathbb{Z}^+}$ denote a full shift space, with the shift map $\sigma : \Sigma \to \Sigma$ defined by $(\sigma \underline{w})_n = w_{n+1}$. For each $n \geq 0$, we define the nth level variation

$$\mathsf{var}_n(\psi) := \sup\{|\psi(x) - \psi(y)| : x_i = y_i, 0 \leq i \leq n-1\}.$$

of $\psi : \Sigma \to \mathbb{R}$. Continuity of the function ψ corresponds to $\mathsf{var}_n(\psi) \to 0$ as $n \to +\infty$.

We say that ψ has *summable variation* if $\sum_{n=0}^{\infty} \mathsf{var}_n(\psi) < +\infty$. Stronger still, we say that ψ is *Hölder continuous* if there exists $0 < \theta < 1$ such that

$$||\psi||_\theta := \sup_{n \geq 1} \left\{ \frac{\mathsf{var}_n(\psi)}{\theta^n} \right\} < +\infty.$$

Given any continuous function $\psi : \Sigma \to \mathbb{R}$ on a subshift of finite type Σ, we can define the *pressure* by

$$P(\psi) = \sup_{\mu \in \mathcal{M}} \left\{ h(\mu) + \int \psi d\mu \right\}$$

where the supremum is over the space $\mathcal{M}$ of all σ-invariant probability measures. This is equivalent to other standard definitions using the variational principle (5).

A measure which realizes this supremum is called an *equilibrium state*. Any continuous function has at least one equilibrium state and every invariant probability measure is the equilibrium state for some continuous potential (cf. (4)). If ψ has summable variation, then there is a unique equilibrium state μ_ψ. Given $x_0, \cdots, x_{n-1} \in \{1, \cdots, k\}$ we can denote

$$[x_0, \cdots, x_{n-1}] = \{\underline{w} \in \Sigma : w_i = x_i \text{ for } 0 \leq i \leq n-1\}.$$

We have the following alternative characterization (cf. (1)).

Lemma 2.1. *If ψ has summable variation then the following are equivalent:*

(1) *A σ-invariant probability measure μ is the unique equilibrium state for ψ; and*

(2) *There exists* $C_1, C_2 > 0$ *such that*

$$C_1 \leq \frac{\mu[w_0, \cdots, w_{n-1}]}{\exp(\psi^n(\underline{w})) - nP(\psi))} \leq C_2 \tag{1}$$

for all $\underline{w} \in \Sigma$.

We shall refer to the unique invariant measure satisfying the Bowen-Gibbs inequality (1) as a *Gibbs measure* (in the sense of Bowen (1)).

Since we are primarily interested in measures, rather than functions, we can replace ψ by $\psi - P(\psi)$ (and still have $\mu_{\psi_1} = \mu_{\psi_1 - P(\psi_1)}$) and so in the sequel we shall assume, without loss of generality, that $P(\psi) = 0$.

Remark 2.2. For the two sided shift shift $\sigma : \Sigma \to \Sigma$ on $\Sigma = \{1, \cdots, k\}^{\mathbb{Z}}$ and a function $\psi : \Sigma \to \mathbb{R}$ we can define $\text{var}_n(\psi) := \sup\{|\psi(x) - \psi(y)| : x_i = y_i, |i| \leq n-1\}$. If $\sum_{n=0}^{\infty} n^2 \text{var}_n(\psi) < +\infty$ we can add a coboundary to obtain a function $\psi'(x)$ which depends only on $(x_n)_{n=0}^{\infty}$ and for which $\sum_{n=0}^{\infty} n \text{var}_n(\psi') < +\infty$. Then Theorem 1.1, for example, applies.

3. Constructing the potential ψ_2

If μ_{ψ_1} is a Gibbs measure for a continuous function ψ_1 (satisfying $P(\psi_1) = 0$) then we can write

$$\nu[z_0, \cdots, z_n] = \sum_{x_0, \cdots, x_n} \mu_{\psi_1}[x_0, \cdots, x_n] \asymp \sum_{x_0, \cdots, x_n} e^{\psi_1^{n+1}(x)}$$

with each summation being over finite strings $x_0, \cdots, x_n$ from Σ_1 for which $\pi(x_i) = z_i$, for $i = 0, \ldots, n$, and any $x \in [x_0, \cdots, x_{n-1}]$. [1] This motivates the construction of the potential function via the limit of a sequence of functions in $C(\Sigma_2, \mathbb{R})$.

We begin by fixing, for the moment, a sequence $\underline{w} \in \Sigma_2$.

Definition 3.1. We would like to consider a sequence of functions $(u_n(\underline{z}))_{n=1}^{\infty}$ in $C(\Sigma_2, \mathbb{R})$ defined by:

$$u_n(\underline{z}) = \frac{\sum_{\underline{x} = x_0 \cdots x_n} \exp(\psi_1^{n+1}(\underline{x}\underline{w}))}{\sum_{\underline{x}' = x_1 \cdots x_n} \exp(\psi_1^{n}(\underline{x}'\underline{w}))} \text{ where } \underline{z} \in \Sigma_2$$

for $n \geq 2$, and $u_1(\underline{z}) = \sum_{x_0} \exp(\psi_1(x_0\underline{w}))$, where:

(1) $\psi_1^n = \sum_{i=0}^{n-1} \psi_1 \circ \sigma^i$ and $\psi_1^{n+1} = \sum_{i=0}^{n} \psi_1 \circ \sigma^i$;
(2) each summation is over finite strings from Σ_1 for which x_i projects to z_i, for $i = 0, \ldots, n$; and

[1]Here $\asymp$ means that the ratio of the two sides are bound above and below (away from zero) independently of n.

(3) $\underline{xw} \in \Sigma_1$ denotes the concaternation of words to give the sequence $(x_0, \cdots, x_n, w_0, w_1 \cdots)$.

It is clear that $u_n(\underline{z})$ depends only on $z_0, \cdots, z_n$, i.e., $u_n : \Sigma_2 \to \mathbb{R}$ is a locally constant function.

The sequence $(u_n(\underline{z}))_{n=1}^{\infty}$ has an explicit dependence on the sequence $\underline{w}$. When appropriate, we will change the notation to reflect the $\underline{w}$ dependence by writing $u_{\underline{w},n}(\underline{z})$. We need the following result.

Proposition 3.2. *The sequence $\{u_{\underline{w},n}(\underline{z})\}$ converges uniformly to a continuous function $u(\underline{z}) > 0$ which is independent of $\underline{w}$.*

We will return to the proof of this proposition in the next section. Before stating a corollary we present a lemma we need in its proof.

Lemma 3.3. *If μ_{ψ_1} is a Gibbs measure with continuous potential ψ_1 then there exists $C > 0$ such that for any $\underline{x} = x_0, \cdots, x_n, \underline{w}, \underline{w}' \in \Sigma_1, n \in \mathbb{N}$, $\dfrac{\exp(\psi_1^{n+1}(\underline{xw}))}{\exp(\psi_1^{n+1}(\underline{xw'}))} \leq C$.*

Proof. By definition, there exists $C_1, C_2 > 0$ such that

$$C_1 \exp(\psi_1^{n+1}(\underline{xw})) \leq \mu_{\psi_1}[x_0, \cdots, x_n] \leq C_2 \exp(\psi_1^{n+1}(\underline{xw'})) \qquad (2)$$

for all $\underline{w} \in \Sigma$, and thus

$$\frac{\exp(\psi_1^{n+1}(\underline{xw}))}{\exp(\psi_1^{n+1}(\underline{xw'}))} \leq \frac{C_2}{C_1}$$

The result follows with $C = C_2/C_1$. $\square$

Corollary 3.4. *The measure $\nu = \Pi_* \mu_{\psi_1}$ is Gibbs measure for $\psi_2(\underline{z}) := \log u(\underline{z})$ (i.e., $\nu = \mu_{\psi_2}$).*

Proof. Fix $n \geq 1$. We can write

$$\begin{aligned} \psi_2^n(\underline{z}) &= \lim_{m \to +\infty} \log u_{\underline{w},m}(\underline{z}) + \cdots + \lim_{m \to +\infty} \log u_{\underline{w},m}(\sigma^n \underline{z}) \\ &= \lim_{m \to +\infty} \log \left(\frac{\sum_{\underline{x}=x_0 \cdots x_m} \exp(\psi_1^{m+1}(\underline{xw}))}{\sum_{\overline{x}=x_{n+1} \cdots x_m} \exp(\psi_1^{m-n}(\overline{x}\underline{w}))} \right). \end{aligned}$$

Moreover, for $m > n$ we can factor the summation

$$\begin{aligned} &\sum_{\underline{x}=x_0 \cdots x_m} \exp(\psi_1^{m+1}(\underline{xw})) \\ &= \sum_{\underline{x}'=x_0 \cdots x_n} \sum_{\overline{x}=x_{n+1} \cdots x_m} \exp(\psi_1^{n+1}(\underline{x}'\overline{x}\underline{w})) \exp(\psi_1^{m-n}(\overline{x}\underline{w})) \end{aligned}$$

and then bound

$$\frac{1}{C}\sum_{\underline{x}'=x_0\cdots x_n}\exp(\psi_1^{n+1}(\underline{x}'\underline{w})) \leq \underbrace{\frac{\sum_{\underline{x}=x_0\cdots x_m}\exp(\psi_1^{m+1}(\underline{x}\underline{w}))}{\sum_{\overline{x}=x_{n+1}\cdots x_m}\exp(\psi_1^{m-n}(\overline{x}\underline{w}))}}_{=\exp(\sum_{i=0}^{n}\log u_{\underline{w},m}(\sigma^i\underline{z}))}$$
$$\leq C\sum_{\underline{x}'=x_0\cdots x_n}\exp(\psi_1^{n+1}(\underline{x}'\underline{w}))$$

where C is as in the previous lemma.

Since μ_{ψ_1} is a Gibbs measure for ψ_1, we can apply (2) and then summing over strings with $\pi(x_i) = z_i$, for $i = 0, \cdots, n-1$, and letting $m \to +\infty$ gives

$$\frac{C_1}{C} \leq \frac{\nu[z_0\cdots z_{n-1}]}{\exp(\psi_2^n(\underline{z}))} \leq C_2 C,$$

It then follows from the definitions that ν is a Gibbs measure for ψ_2. $\square$

4. Proof of Proposition 3.2

We return to the proof postponed from the previous section.

Definition 4.1. Let $s > n$, $\overline{x} = x_{n+1}\cdots x_s$ be a given finite string and $\underline{w} \in \Sigma$. We define

$$P^{(s,n)}(\overline{x},\underline{w}) = \frac{\sum_{\underline{x}=x_1\cdots x_n}\exp(\psi_1^n(\underline{x}\overline{x}\underline{w}))}{\sum_{\underline{x}'=x_1\cdots x_s}\exp(\psi_1^n(\underline{x}'\underline{w}))}$$

(where $\pi(x_i) = z_i$ for $i = 1, \cdots, s$).

We require the following lemma.

Lemma 4.2. *For $s > n$ we have the identity*

$$u_{\underline{w},s}(\underline{z}) = \sum_{\overline{x}=x_{n+1}\cdots x_s} u_{\overline{x}\underline{w},n}(\underline{z})P^{(s,n)}(\overline{x},\underline{w})$$

where $\underline{z} \in \Sigma_2$ (and $\pi(x_i) = z_i$ for $i = n+1, \cdots, s$).

Proof. By definition, the *numerator* of $u_{\underline{w},s}(\underline{z})$ is

$$\sum_{\underline{x}=x_0\cdots x_s}\exp(\psi_1^{s+1}(\underline{x}\underline{w}))$$
$$= \sum_{\underline{x}=x_0\cdots x_n}\sum_{\overline{x}=x_{n+1}\cdots x_s}\exp(\psi_1^{n+1}(\underline{x}\overline{x}\underline{w}))\exp(\psi_1^{s-n}(\overline{x}\underline{w})).$$

(where $\pi(x_i) = z_i$ for $i = 0, \dots, s$) and we have used $\psi_1^{s+1}(\overline{x}\underline{x}\underline{w}) = \psi_1^{n+1}(\underline{x}\overline{x}\underline{w}) + \psi_1^{s-n}(\overline{x}\underline{w})$. We can further rewrite this as

$$\sum_{\overline{x}=x_{n+1}\cdots x_s} \underbrace{\left(\frac{\sum_{\underline{x}=x_0\cdots x_n} \exp(\psi_1^{n+1}(\underline{x}\overline{x}\underline{w}))}{\sum_{\underline{x}'=x_1\cdots x_n} \exp(\psi_1^{n}(\underline{x}'\overline{x}\underline{w}))}\right)}_{u_{\overline{x}\underline{w},n}(\underline{z})} \underbrace{\left(\sum_{\underline{x}'=x_1\cdots x_n} \exp(\psi_1^{n}(\underline{x}'\overline{x}\underline{w}))\right) \exp(\psi_1^{s-n}(\overline{x}\underline{w}))}_{\sum_{\underline{x}'=x_1\cdots x_n} \exp(\psi_1^{s+1}(\underline{x}'\overline{x}\underline{w}))}$$

We can now divide by the *denominator* of $u_s(\underline{z})$ to get the result. $\square$

Corollary 4.3. *We can write*

$$\frac{u_{\underline{w},s}(\underline{z})}{u_{\underline{w}',s}(\underline{z})} = \frac{\sum_{\overline{x}=x_{n+1}\cdots x_s} u_{\overline{x}\underline{w},n}(\underline{z}) P^{(s,n)}(\overline{x}, \underline{w})}{\sum_{\overline{x}=x_{n+1}\cdots x_s} u_{\overline{x}\underline{w}',n}(\underline{z}) P^{(s,n)}(\overline{x}, \underline{w}')}.$$

The following special case is illustrative.

Example 4.4 (Markov case)**.** Assume ψ_1 depends on only the first two coordinates. Let $s = n+2$ and then observe from the definitions that $u_{x_n x_{n+1}\underline{w},n}(\underline{z})$ is independent of $\underline{w}$. In particular,

$$\frac{u_{w_1 w_2,n+2}(\underline{z})}{u_{w_1' w_2',n+2}(\underline{z})} = \frac{\sum_{x_n,x_{n+1}} u_{x_n x_{n+1},n}(\underline{z}) P^{(n+2,n)}(x_n x_{n+1}, w_1 w_2)}{\sum_{x_n,x_{n+1}} u_{x_n x_{n+1},n}(\underline{z}) P^{(n+2,n)}(x_n x_{n+1}, w_1' w_2')}$$

In this case Lemma 4.2 corresponds to a linear action by a strictly positive matrix, which contracts the simplex and so converges at an exponential rate to a fixed line.

It remains to use the corollary to complete the proof of Proposition 3.2. We require the following.

Lemma 4.5. *There exists $c > 0$ such that $\frac{P^{(s,n)}(\overline{x},\underline{w})}{P^{(s,n)}(\overline{x},\underline{w}')} \geq c$ for all $\overline{x} = x_n \cdots x_s$ and $\underline{w}, \underline{w}' \in \Sigma$.*

Proof. Since $\underline{x}\overline{x}\underline{w}$ and $\underline{x}\overline{x}\underline{w}'$ agree in s places we see that

$$|\psi_1^n(\underline{x}\overline{x}\underline{w}) - \psi_1^n(\underline{x}\overline{x}\underline{w}')| \leq \log\left(\frac{C_2}{C_1}\right)$$

and the result follows with $c = \left(\frac{C_1}{C_2}\right)^2$. $\square$

For each $n \geq 0$ and $\underline{z} \in \Sigma_2$ we denote

$$\lambda_n = \lambda_n(\underline{z}) := \sup\left\{\frac{u_{\underline{w},n}(\underline{z})}{u_{\underline{w}',n}(\underline{z})} : \underline{w}, \underline{w}' \in \Sigma_2\right\} \geq 1.$$

Lemma 4.6. *The sequence $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots \geq 1$ is a monotone decreasing sequence. Furthermore, the intervals* $[\inf_{\underline{w}} u_{\underline{w},n}(\underline{z}), \sup_{\underline{w}} u_{\underline{w},n}(\underline{z})]$ *form a nested sequence in n.*

Proof. Fix $n \geq 1$. By definition

$$u_{\underline{w},n+1}(\underline{z}) = \frac{\sum_{\underline{x}=x_0\cdots x_n}\sum_{x_{n+1}}\exp(\psi_1^{n+1}(\underline{x}x_{n+1}\underline{w})\exp(\psi_1(x_{n+1}\underline{w}))}{\sum_{\underline{x}'=x_1\cdots x_n}\sum_{x_{n+1}}\exp(\psi_1^{n}(\underline{x}'x_{n+1}\underline{w})\exp(\psi_1(x_{n+1}\underline{w}))}$$
$$\leq \max_{x_{n+1}} \underbrace{\frac{\sum_{\underline{x}=x_0\cdots x_n}\exp(\psi_1^{n+1}(\underline{x}x_{n+1}\underline{w})}{\sum_{\underline{x}'=x_1\cdots x_n}\exp(\psi_1^{n}(\underline{x}'x_{n+1}\underline{w})}}_{u_{x_{n+1}\underline{w},n}(\underline{z})}$$

The inequality here comes from the fact that

$$\frac{\sum_{x_{n+1}} a(x_{n+1})}{\sum_{x_{n+1}} b(x_{n+1})} \leq \max_{x_{n+1}} \frac{a(x_{n+1})}{b(x_{n+1})}.$$

Similarly, $\min_{x'_{n+1}} u_{x'_{n+1}\underline{w},n}(\underline{z}) \leq u_{\underline{w},n+1}(\underline{z})$. Thus if $\underline{w}, \underline{w}' \in \Sigma_2$ then

$$\frac{u_{\underline{w},n+1}(\underline{z})}{u_{\underline{w}',n+1}(\underline{z})} \leq \frac{u_{x_n\underline{w},n}(\underline{z})}{u_{x'_n\underline{w}',n}(\underline{z})} \leq \lambda_n.$$

Taking the supremum over $\underline{w}, \underline{w}' \in \Sigma_2$ gives that $\lambda_{n+1} \leq \lambda_n$. □

The following inequality is central to our analysis.

Lemma 4.7. *Let $s > n$. Then*

$$\lambda_s \leq c\exp\left(\sum_{k=s-n}^{s} var_k(\psi_1)\right) + (1-c)\lambda_n \tag{3}$$

Proof. If $\lambda_n \leq \exp\left(\sum_{k=s-n}^{s} \mathrm{var}_k(\psi_1)\right)$ then

$$\lambda_s = c.\lambda_s + (1-c)\lambda_s \leq c\exp\left(\sum_{k=s-n}^{s} \mathrm{var}_k(\psi_1)\right) + (1-c)\lambda_n$$

as required. Therefore, we can assume that $\lambda_n \geq \exp\left(\sum_{k=s-n}^{s} \mathrm{var}_k(\psi_1)\right)$.

For ease of notation, we denote $b_{s,n} := \exp\left(\sum_{k=s-n}^{s} \mathrm{var}_k(\psi_1)\right)$. We want to combine Corollary 4.3 and Lemma 4.6. For any $\underline{w}, \underline{w}' \in \Sigma$ we can bound

$$
\begin{aligned}
&\frac{u_{\underline{w},s}(\underline{z})}{u_{\underline{w}',s}(\underline{z})}\\
&= \frac{\sum_{\overline{x}=x_{n+1}\cdots x_s} c u_{\overline{x}\underline{w},n}(\underline{z}) P^{(s,n)}(\overline{x},\underline{w}) + (1-c) u_{\overline{x}\underline{w},n}(\underline{z}) P^{(s,n)}(\overline{x},\underline{w})}{\sum_{\overline{x}=x_{n+1}\cdots x_s} c u_{\overline{x}\underline{w}',n}(\underline{z}) P^{(s,n)}(\overline{x},\underline{w}) + (P^{(s,n)}(\overline{x},\underline{w}') - cP^{(s,n)}(\overline{x},\underline{w})) u_{\overline{x}\underline{w}',n}(\underline{z})}\\
&\le \frac{\sum_{\overline{x}=x_{n+1}\cdots x_s} c \left(b_{s,n} \min_{\overline{x}'}\{u_{\overline{x}'\underline{w}',n}(\underline{z})\}\right) P^{(s,n)}(\overline{x},\underline{w}) + (1-c)\max_{\overline{x}}\{u_{\overline{x}\underline{w},n}(\underline{z})\} P^{(s,n)}(\overline{x},\underline{w})}{\sum_{\overline{x}=x_{n+1}\cdots x_s} c \min_{\overline{x}'}\{u_{\overline{x}'\underline{w}',n}(\underline{z})\} P^{(s,n)}(\overline{x},\underline{w}) + (P^{(s,n)}(\overline{x},\underline{w}') - cP^{(s,n)}(\overline{x},\underline{w})) \min_{\overline{x}'}\{u_{\overline{x}'\underline{w}',n}(\underline{z})\}}\\
&= \frac{\sum_{\overline{x}=x_{n+1}\cdots x_s} c b_{s,n} P^{(s,n)}(\overline{x},\underline{w}) + (1-c)\frac{\max_{\overline{x}}\{u_{\overline{x}\underline{w},n}(\underline{z})\}}{\min_{\overline{x}'}\{u_{\overline{x}'\underline{w}',n}(\underline{z})\}} P^{(s,n)}(\overline{x},\underline{w})}{\sum_{\overline{x}=x_{n+1}\cdots x_s} c P^{(s,n)}(\overline{x},\underline{w}) + (P^{(s,n)}(\overline{x},\underline{w}') - cP^{(s,n)}(\overline{x},\underline{w}))}\\
&\le c b_{s,n} + (1-c)\frac{\max_{\overline{x}}\{u_{\overline{x}\underline{w},n}(\underline{z})\}}{\min_{\overline{x}'}\{u_{\overline{x}'\underline{w}',n}(\underline{z})\}}\\
&\le c b_{s,n} + (1-c)\lambda_n
\end{aligned}
$$

as required. For the second line here we decreased the denominator by replacing $u_{\overline{x}\underline{w}',n}(\underline{z})$ with $\min_{\overline{x}'}\{u_{\overline{x}'\underline{w}',n}(\underline{z})\}$ and increased the numerator by replacing $u_{\overline{x}\underline{w},n}(\underline{z})$ with $\max_{\overline{x}}\{u_{\overline{x}\underline{w},n}(\underline{z})\}$ and $b_{s,n}\min_{\overline{x}'}\{u_{\overline{x}'\underline{w}',n}(\underline{z})\}$. □

Choosing $s = 2n$ gives the following.

Corollary 4.8. *For any $n \ge 1$ we can bound*

$$\lambda_{2n} \le \underbrace{c\exp\left(\sum_{k=n}^{2n} var_k(\psi_1)\right)}_{b_{2n,n}} + (1-c)\lambda_n$$

In particular, since $b_{2n,n} \to 1$ as $n \to +\infty$ it is clear that $\lim_{n\to+\infty}\lambda_n = 1$. Moreover, the regularity of u, and thus ψ_2, is determined by the speed of convergence.

To complete the proof of Proposition 3.2 we need the following simple lemma.

Lemma 4.9. *We have that*

(1) *for any $\underline{w} \in \Sigma_2$ the limit $u(\underline{z}) = \lim_{n\to+\infty} u_{\underline{w},n}(\underline{z})$ exists; and*
(2) *$\psi_2 := \log(u(\underline{z}))$ is continuous in $\underline{z} \in \Sigma_2$.*

Proof. For the first part, it suffices to use the previous observation that $\lim_{n\to+\infty}\lambda_n = 1$.

For the second part we observe that if $\underline{z}_1, \underline{z}_2 \in \Sigma_2$ agree in n terms then for any $\underline{w}$, we have $u_n(\underline{z}_1) = u_n(\underline{z}_2)$. Thus

$$\left|\frac{u(\underline{z}_1)}{u(\underline{z}_2)}\right| \leq \sup\left\{\frac{u_{\underline{w},n}(\underline{z}_1)}{u_{\underline{w}'n}(\underline{z}_2)} : \underline{w}, \underline{w}' \in \Sigma_2\right\} = \lambda_n$$

and again the result follows from $\lim_{n\to+\infty} \lambda_n = 1$. □

We now have that $\psi_2 = \log u$ is a well defined continuous function which is a potential for ν, which completes the proof of Proposition 3.2.

Remark 4.10. There are more general hypotheses on the hypotheses that one might consider including , for example, the Walters' condition. However, it is not immediately clear how the methods in this paper can be adapted to that case. However, observe that any potential ψ_1 satisfying the Bowen-Gibbs inequality must necessarily be uniformly continuous. Since the only condition on ψ_1 that we have used is that it is uniformly continuous, Proposition 3.2 shows that if μ_{ψ_1} is a Gibbs measure. Then $\nu := \Pi(\mu_{\psi_1})$ is a Gibbs measure.

5. Regularity of the potential ψ_2

In this section we will consider the regularity properties of $\psi_2 := \log u$. The following is our main result.

Theorem 5.1. *Let $\kappa \geq 0$. If $\sum_{n=0}^{\infty} n^{\kappa+1} var_n(\psi_1) < \infty$ then $\sum_{n=0}^{\infty} n^{\kappa} var_n(\psi_2) < \infty$.*

Proof. Let $0 < c < 1$ be as in Lemma 4.5. Choose $1 < \beta < 1/(1-c)$ and an integer $M > 1$ sufficiently large that $\alpha := \beta(1-c)\left(1 + \frac{1}{M}\right)\left(1 - \frac{1}{M}\right)^{-\kappa} < 1$. Let us denote $a_n = \log \lambda_n$ and recall the trivial inequality $1 + x \leq \exp(x) \leq 1 + \beta x$, for $x > 0$ sufficiently small. Thus providing N_0 is sufficiently large we can deduce from (3) that for any $n > N_0$

$$\begin{aligned} 1 + a_n \leq \exp(a_n) &\leq c.\exp\left(\sum_{m=[n/M]}^{n} var_m(\psi_1)\right) + (1-c)\exp(a_{n-[n/M]}) \\ &\leq c + (1-c) + \beta c \sum_{m=[n/M]}^{n} var_m(\psi_1) + \beta(1-c) a_{n-[n/M]} \end{aligned}$$

and hence that for any $N > N_0$,

$$\sum_{n=N_0}^{N} n^{\kappa} a_n \leq \beta c \sum_{n=N_0}^{N} n^{\kappa} \sum_{m=[n/M]}^{n} var_m(\psi_1) + \beta(1-c) \sum_{n=N_0}^{N} n^{\kappa} a_{n-[n/M]}$$

(where $[\cdot]$ denotes the integer part).

We can bound

$$\begin{aligned}\sum_{n=N_0}^{N} n^\kappa \sum_{m=[n/M]}^{n} var_m(\psi_1) &\leq M^\kappa \sum_{n=N_0}^{N} \sum_{m=[n/M]}^{n} m^\kappa var_m(\psi_1) \\ &\leq M^{\kappa+1} \sum_{n=N_0}^{N} n^{\kappa+1} var_n(\psi_1)\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=N_0}^{N} n^\kappa a_{n-[n/M]} \\ &\leq \frac{1}{\left(1-\frac{1}{M}\right)^\kappa} \sum_{n=N_0}^{N} (n-[n/M])^\kappa a_{n-[n/M]} \\ &\leq \frac{1}{\left(1-\frac{1}{M}\right)^\kappa} \left(\sum_{m=\left[N_0-\frac{N_0}{M}\right]}^{\left[N-\frac{N}{M}\right]+1} m^\kappa a_m + \sum_{\substack{N_0 \leq n \leq N \\ M \mid n+1}} (n-[n/M])^\kappa a_{n-[n/M]} \right) \\ &\leq \frac{\left(1+\frac{1}{M}\right)}{\left(1-\frac{1}{M}\right)^\kappa} \sum_{m=N_0}^{N} m^\kappa a_m + O(1)\end{aligned}$$

where we have used that

$$\sum_{\substack{N_0 \leq n \leq N \\ M \mid n+1}} (n-[n/M])^\kappa a_{n-[n/M]} \leq \frac{1}{M} \sum_{m=N_0-[N_0/M]}^{N} m^k a_m$$

$$\text{and} \quad \sum_{m=N_0-[N_0/M]}^{N_0-1} m^k a_m = O(1).$$

Comparing the above inequalities we can bound

$$\underbrace{\left(1-\beta(1-c)\frac{\left(1+\frac{1}{M}\right)}{\left(1-\frac{1}{M}\right)^\kappa}\right)}_{>0} \sum_{n=N_0}^{N} n^\kappa a_n \leq \beta c M^{\kappa+1} \sum_{n=N_0}^{N} n^{\kappa+1} var_n(\psi_1) + O(1)$$

Letting $N \to +\infty$ we see that $\sum_{n=N_0}^{\infty} n^\kappa a_n < \infty$, which completes the proof. $\square$

When $\kappa = 0$ we have the following corollary, which was stated in the Introduction.

Corollary 5.2. *If* $\sum_{n=0}^{\infty} nvar_n(\psi_1) < \infty$ *then* $\sum_{n=0}^{\infty} var_n(\psi_2) < \infty$.

Another application of (3) is the following.

Theorem 5.3. *Assume there exists* $c_1 > 0$ *and* $0 < \theta_1 < 1$ *such that* $var_n(\psi_1) \le c_1\theta_1^{\sqrt{n}}$, *for all* $n \ge 0$ *then there exists* $c_2 > 0$ *and* $0 < \theta_2 < 1$ *such that* $var_n(\psi_2) \le c_2\theta_2^{\sqrt{n}}$, *for all* $n \ge 0$

Proof. By inequality (3) we can write

$$\begin{aligned}
\lambda_n &\le c\exp\left(c_1 \sum_{k=n-[\sqrt{n}]}^{n} \theta_1^{\sqrt{k}}\right) + (1-c)\lambda_{n-[\sqrt{n}]} \\
&\le c\exp\left(C\theta^{[\sqrt{n}]}\right) + (1-c)\lambda_{n-[\sqrt{n}]}
\end{aligned}$$

for any $\theta_1 < \theta < 1$ and some $C > 0$. Using this inequality inductively $[\sqrt{n}]$ times, we can write

$$\begin{aligned}
\lambda_n &\le c\exp\left(C\theta^{[\sqrt{n}]}\right) + (1-c)\left(c\exp\left(C\theta^{[\sqrt{n}]}\right) + (1-c)\lambda_{n-2[\sqrt{n}]}\right) \\
&\quad \cdots \\
&\le c\exp\left(C\theta^{[\sqrt{n}]}\right)\sum_{k=0}^{[\sqrt{n}]}(1-c)^k + (1-c)^{[\sqrt{n}]}\lambda_{n-[\sqrt{n}]^2} \\
&\le \exp\left(C\theta^{[\sqrt{n}]}\right) + (1-c)^{[\sqrt{n}]}\lambda_0.
\end{aligned}$$

In particular, we see that $|\lambda_n - 1| = O\left(\theta_2^{\sqrt{n}}\right)$, where $\theta_2 = \max\{\theta, (1-c)\}$ from which the result follows. □

The following is an easy consequence of the theorem and its proof.

Corollary 5.4. *Assume there exists* $c_1 > 0$ *and* $0 < \theta < 1$ *such that* $var_n(\psi_1) \le c_1\theta^n$ *for all* $n \ge 0$ *(i.e.,* ψ_1 *is Hölder continuous) then there exists* $c_2 > 0$ *such that* $var_n(\psi_2) \le c_2\theta^{\sqrt{n}}$ *for all* $n \ge 0$.

The same conclusion, using a different proof, appears in (3).

References

[1] R. Bowen. *Ergodic Theory of Axiom A diffeomorphisms.* Lecture Notes of Mathematics, Springer-Verlag, Berlin, 1975.

[2] J.-R. Chazottes and E. Ugalde, Projection of Markov Measures May Be Gibbsian, J. Statist. Phys. 111 (2003), no. 5-6, 1245–1272.

[3] J.R. Chazottes and E. Ugalde: Preservation Of Gibbsianness Under Amalgamation Of Symbols, *preprint.*

[4] D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, New York, 1978.

[5] P.Walters: *Ergodic Theory*, Graduate Texts in Mathematics 79, Springer-Verlag, Berlin, 1982.

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